Announcements

1) The norm in problem #2 is the norm from the inner product:

$$\|v\| = \langle v, v \rangle^{1/2}$$

Gram-Schmidt Theorem:

Let V be an inner product space and let $S \subseteq V$ be a set of nonzero, linearly independent vectors in V, and $|S|$ is either finite or countably infinite.

Writing $S = \{w_1, w_2, \cdots\}$

with possible termination,

set $x_1 = w_1$ and $\forall$

$n > 1$, set

$$\boxed{x_n = w_n - \sum_{i=1}^{n-1} \frac{\langle w_n, x_i \rangle}{\|x_i\|_2^2} x_i}$$

Then $\{x_n\}$ is orthogonal and

$$\operatorname{span}\left(\{x_n\}_{n=1}^{\infty}\right) = \operatorname{span}(S)$$

if S is countable,

and if $|S| = k < \infty$,

$$\operatorname{span}\left(\{x_n\}_{n=1}^{k}\right) = \operatorname{span}(S)$$

Proof: By induction.

Given $S = \{w_1, w_2, \dots, \}$,

Set $x_1 = w_1$,

$$x_2 = w_2 - \frac{\langle w_2, x_1 \rangle}{\|x_1\|_2^2} x_1 .$$

Then $\langle x_1, x_2 \rangle$

$$= \left\langle x_1,\ w_2 - \frac{\langle w_2, x_1 \rangle}{\|x_1\|_2^2} x_1 \right\rangle$$

$$= \langle x_1, w_2 \rangle - \left\langle x_1, \frac{\langle w_2, x_1 \rangle}{\|x_1\|_2^2} x_1 \right\rangle$$

$$= \langle x_1, w_2 \rangle - \frac{\overline{\langle w_2, x_1 \rangle}}{\cancel{\|x_1\|_2^2}} \cancel{\|x_1\|_2^2}$$

$$= \langle x_1, w_2 \rangle - \langle x_1, w_2 \rangle = 0$$

Now assume $n > 2$ and that we have chosen $\{x_1, x_2, \ldots, x_{n-1}\}$ as indicated in the statement of the theorem and $\langle x_i, x_j \rangle = 0 \ \forall \, i \neq j$, $1 \leq i, j \leq n-1$.

Let $$x_n = w_n - \sum_{i=1}^{n-1} \frac{\langle w_n, x_i \rangle}{\|x_i\|_2^2} x_i$$

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Then if $1 \leq j \leq n-1$,

$$\langle x_j, x_n \rangle =$$

$$\langle x_j, w_n - \sum_{i=1}^{n-1} \frac{\langle w_n, x_i \rangle}{\|x_i\|_2^2} x_i \rangle$$

$$= \langle x_j, w_n \rangle - \sum_{i=1}^{n-1} \frac{\overline{\langle w_n, x_i \rangle \langle x_j, x_i \rangle}}{\|x_i\|_2^2}$$

$$= \langle x_j, w_n \rangle - \sum_{i=1}^{n-1} \frac{\overline{\langle w_n, x_i \rangle} \langle x_j, x_i \rangle}{\|x_i\|_2^2}$$

But by assumption, $\langle x_j, x_i \rangle$ is zero for $i \neq j$.

The sum then reduces to

$$\langle x_j, w_n \rangle - \frac{\overline{\langle w_n, x_j \rangle} \|x_j\|_2^2}{\|x_j\|_2^2}$$

We know $x_j \neq 0_V$,

for if it was, then

$$0_V = x_j = w_j - \sum_{i=1}^{j-1} \frac{\langle w_j, x_i \rangle x_i}{\|x_i\|_2^2}$$

$$\Rightarrow w_j = \sum_{i=1}^{j-1} \frac{\langle w_j, x_i \rangle}{\|x_i\|_2^2} x_i$$

and since each previous x_i is a linear combination of w_t for $1 \leq t < i$, then w_j would be a linear combination of w_i's for $1 \leq i < j$, contradicting the linear independence of S.

Therefore, $x_j \neq 0_V$, so

$$\langle x_j, w_n \rangle - \frac{\overline{\langle w_n, x_j \rangle}}{\cancel{\|x_j\|_2^2}} \cancel{\|x_j\|_2^2}$$

$$= \langle x_j, w_n \rangle - \langle x_j, w_n \rangle = 0$$

To show that the span of the x_n's is the span of S, note that we have already observed that each x_n is a linear combination of w_i's for $1 \leq i \leq n$.

Therefore $\text{span}(\{x_n\})$

$$\subseteq \text{span}(S).$$

But since

$$x_n = w_n - \sum_{i=1}^{n-1} \frac{\langle w_n, x_i \rangle}{\|x_i\|_2^2} x_i \,,$$

$$w_n = x_n + \sum_{i=1}^{n-1} \frac{\langle w_n, x_i \rangle}{\|x_i\|_2^2} x_i$$

$\Rightarrow \quad w_n \in \operatorname{span}(\{x_n\})$.

Therefore, $\operatorname{span}(\{x_n\})$

$= \operatorname{span}(S)$. $\square$

Definition: (orthogonal complement)

If V is an inner product space and $S \subseteq V$ is any nonempty set, define the orthogonal complement $S^\perp$ of S in V to be

$$S^\perp = \{ w \in V \mid \langle w, v \rangle = 0 \ \forall v \in S \}$$

Example 1: (one vector in $\mathbb{R}^3$)

$(1,2,3) = v \in \mathbb{R}^3$.

Let's figure out

$\{v\}^{\perp}$.

Let $w = (-2,1,0)$.

$\langle v, w \rangle = 1 \cdot (-2) + 2 \cdot 1 + 0$

$= 0$

So $w \in \{v\}^\perp$.

Without Gram-Schmidt available, we could then let $u = v \times w$.

Then $u \in \{v\}^\perp$, but moreover, $u \in \{w\}^\perp$.

So $\{v, u, w\}$ is an orthogonal set in $\mathbb{R}^3$, hence a basis for $\mathbb{R}^3$.

So if $x \in \mathbb{R}^3$, $\exists$

$\alpha, \beta, \gamma \in \mathbb{R}$ with

$$x = \alpha v + \beta u + \gamma w.$$

If $x \in \{v\}^\perp$, then

$$\begin{aligned}
0 &= \langle x, v \rangle \\
&= \langle \alpha v + \beta u + \gamma w, v \rangle \\
&= \alpha \langle v, v \rangle + \beta \cancel{\langle u, v \rangle} + \gamma \cancel{\langle w, v \rangle} \\
&= \alpha \|v\|_2^2 .
\end{aligned}$$

zero

$v \neq (0,0,0)$, so

$\alpha = 0$ and

$x = \beta u + \gamma w$.

This says that $\{v\}^{\perp}$ is a plane in $\mathbb{R}^3$ that passes through the origin and is orthogonal to the line through the origin determined by v.

This shows $\{v\}^{\perp}$

is a subspace

of $\mathbb{R}^3$.